Exponential Growth

Many quantities grow or decay at a rate proportional to their size. A quantity y that grows or decays at a rate proportional to its size fits in an equation of the form

$$\boxed{\frac{dy}{dt} = ky.}$$

- This is a special example of a **differential equation** because it gives a relationship between a function and one or more of its derivatives.
- If $k < 0$, the above equation is called **the law of natural decay** and if $k > 0$, the equation is called **the law of natural growth.**
- A solution to a differential equation is a function y which satisfies the equation.

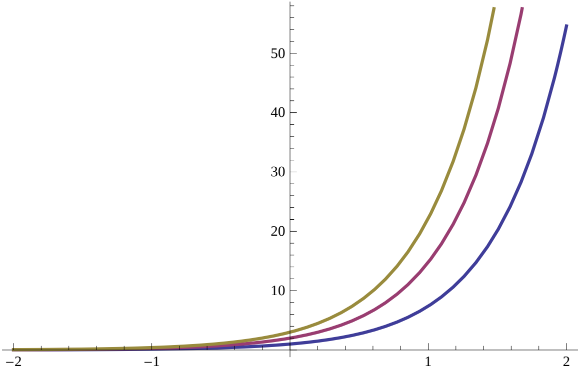
Solutions to the Differential Equation $\frac{dy(t)}{dt} = ky(t)$

It is not difficult to see that $y(t) = e^{kt}$ is one solution to the differential equation $\frac{dy(t)}{dt} = ky(t)$.

- as with antiderivatives, the above differential equation has many solutions.
- In fact any function of the form
$$y(t) = Ce^{kt}$$
is a solution for any constant C.
- We will prove later that every solution to the differential equation above has the form $y(t) = Ce^{kt}$.
- Setting $t = 0$, we get
The **only solutions to the differential equation** $dy/dt = ky$ are the exponential functions
$$\boxed{y(t) = y(0)e^{kt}}$$

Solutions to the Differential Equation $\frac{dy(t)}{dt} = 2y(t)$

Here is a picture of three solutions to the differential equation $dy/dt = 2y$, each with a different value $y(0)$.



We see that each one "starts" with a different initial value $y(0)$.

Population Growth

Population Growth Let P be the size of a population at time t. The law of natural growth is a good model for population growth (up to a certain point):

$$\boxed{\frac{dP}{dt} = kP \quad \text{and} \quad P(t) = P(0)e^{kt}}$$

Note that the relative growth rate, $\frac{dP}{dt}/P = k$ is constant.

Population Growth: Example 1

Population GrowthExample The population of Mathland at the end the year 2000 was 500. The population increases (continuously or steadily) by approximately 10% per year. What is the function $P(t)$, the size of the population after t years, using the exponential model above?

- What differential equation does the function $P(t)$ satisfy? $\frac{dP(t)}{dt} = kP(t)$
- What is the value of k? k = rate of growth = 0.1
- What is $P(0)$? P(0) = initial pop. size = 500
- Give a formula for $P(t)$ $P(t) = P(0)e^{kt} = 500e^{0.1t}$
- What will the population be at the end of the year 2050? At the end of the year 2050 we will have $t = 50$ and the population will be

$$P(50) = 500e^{0.1(50)} = 500e^{5} \approx 74,206$$

Population Growth: Example 2

Example The population of Calculand was 700 in the year 2000 and was 3000 in the year 2010. Using the exponential model for population growth, find an estimate for the population of Calculand in 2015.

- $P(0) = ?$ If we set $t = 0$ in the year 2000, therefore $P(0) = 700$
- $P(10) = ?$ When $t = 10$ the year is 2010, so $P(10) = 3000$.
- Find the value of k. $P(t) = P(0)e^{kt}$ so $P(10) = 700e^{10k}$. Therefore $3000 = 700e^{10k}$. To solve for k, we apply the logarithm to both sides to get
$$\ln(3000) = \ln(700) + \ln(e^{10k}) = \ln(700) + 10k.$$
Therefore $k = \dfrac{\ln(3000) - \ln(700)}{10} = \dfrac{\ln(30/7)}{10} \approx 0.147.$
- Find the formula for $P(t)$ and use it to find $P(15)$. We now have
$$P(t) = P(0)e^{kt} = 700e^{0.147t}.$$
To find the population in 2015, we find $P(15) = 700e^{0.147 \times 15} \approx 6,210.$

Radioactive Decay

Radioactive Decay Radioactive substances decay at a rate proportional to their mass.

$$\boxed{\frac{dm}{dt} = km \quad \text{and} \quad m(t) = m_0 e^{kt},}$$

where $m(t)$ denotes the mass of the substance at time t and m_0 denotes the mass of the substance at time $t = 0$. **The half-life of a radioactive substance is the time required for half of the quantity to decay.**

Carbon Dating the haf-life of Carbon-14 is approximately $t_{1/2} = 5,730$ years (there is some variety in this depending on variables such as location). When a plant or animal dies, it stops taking in Carbon and the carbon it contains starts to decay. We can use this to figure out the age of artifacts by estimating the original mass of Carbon-14 in the object and the amount at present. We use the half-life to find the value of k above.

- **Example** A bowl made of oak has about 40% of the carbon-14 that a similar quantity of living oak has today. Estimate the age of the bowl.

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- $m(t) = m(0)e^{kt}$ where $m(t)$ is the amount of carbon in the bowl t years after it was made.
- To find k, we use the half-life of Carbon-14:

$$\frac{m(5,730)}{m(0)} = \frac{1}{2} = \frac{m(0)e^{5730k}}{m(0)} = e^{5730k}.$$

- Applying the natural logarithm, we get $\ln(\frac{1}{2}) = \ln(e^{5730k}) = 5730k$ giving us that $k = \frac{\ln(\frac{1}{2})}{5730}$.
- To find the age we solve for the time t when the Carbon-14 had decayed to 40% of its original value. We solve for t is $\frac{m(t)}{m(0)} = \frac{m(0)e^{kt}}{m(0)} = .4$
- that is

$$e^{kt} = .4 \quad \text{or} \quad \ln(e^{kt}) = \ln(.4) \quad \text{or} \quad kt = \ln(.4)$$

- This gives $t = \frac{\ln(.4)}{\frac{\ln(\frac{1}{2})}{5730}} \approx 7575$ years.
- The formula used for reference by scientists $t = \frac{\ln(M/M_0)}{\ln(1/2)} t_{1/2}$.

Compund Interest

This differential equation also applies to interest compunded continuously

$$\frac{dA(t)}{dt} = rA(t)$$

$A(t)$ = amount in account at time t, r = interest rate (see below) **Interest** If we invest \$$A_0$ in an account paying $r \times 100$ % interest per anumn and the interest is compounded continuously, the amount in the account after t years is given by

$$\boxed{A(t) = A_0 e^{rt}.}$$

Interest Compounded Continuously

Example If I invest \$1000 for 5 years at a 4% interest rate with the interest compounded continuously,

(a) how much will be in my account at the end of the 5 years?

- *We are given that* $A_0 = 1000$ *and* $r = 0.04$.
- *Because the interest is compounded continuously, we have* $A(t) = A_0 e^{0.04t} = 1000e^{0.04t}$
- $A(5) = 1000e^{0.04(5)} = \$1221.4.$

(b) How long before there is \$2000 in the account?

- *We must solve for* t *in the equation* $2000 = 1000e^{0.04t}$.
- *Dividing by 1000 and taking the natural logarithm of both sides, we get*

$$2 = e^{0.04t} \quad \rightarrow \quad \ln 2 = 0.04t \quad \rightarrow \quad t = \ln 2/0.04 \approx 17.33 yrs.$$

Interest compounded n times per year

Sometimes interest is not compounded continuously. If I invest $\$A_0$ in an account with an interest rate of $r \times 100\%$ per annum, the amount in the bank account after t years depends on the number of times the interest is compounded per year. In the chart below

$A_0 = A(0)$ is the initial amount invested at time $t = 0$.
$A(t)$ is the amount in the account after t years.
$n =$ the number of times the interest is compounded per year.

We Have $$\boxed{A(t) = A_0(1 + \frac{r}{n})^{nt}}$$

Amt. after t years	$A(0)$	$A(1)$	$A(2)$	...	$A(t)$
$n = 1$	A_0	$A_0(1+r)$	$A_0(1+r)^2$	...	$A_0(1+r)^t$
$n = 2$	A_0	$A_0(1+\frac{r}{2})^2$	$A_0(1+\frac{r}{2})^4$	...	$A_0(1+\frac{r}{2})^{2t}$
$n = 12$	A_0	$A_0(1+\frac{r}{12})^{12}$	$A_0(1+\frac{r}{12})^{24}$	...	$A_0(1+\frac{r}{12})^{12t}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	A_0	$A_0(1+\frac{r}{n})^n$	$A_0(1+\frac{r}{n})^{2n}$	...	$A_0(1+\frac{r}{n})^{nt}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$n \to \infty$	A_0	$\lim_{n\to\infty} A_0(1+\frac{r}{n})^n$	$\lim_{n\to\infty} A_0(1+\frac{r}{n})^{2n}$	...	$\lim_{n\to\infty} A_0(1+\frac{r}{n})^{nt}$
(compounded continuously)	$= A_0$	$= A_0 e^r$	$= A_0 e^{2r}$	...	$= A_0 e^{rt}$

Examples

Example If I borrow $50,000 at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?

- $A(t) = A_0(1 + \frac{r}{n})^{nt}$
- $A(t) = 50,000(1 + \frac{.1}{4})^{4t}$
- $A(5) = 50,000(1 + \frac{.1}{4})^{20} \approx 81,930.82$